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Quantity Conjectural Variations in Oligopoly Games under Different Demand and Cost Functions and Multilevel Leadership

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Abstract—This paper considers a noncooperative game of quantity competition among firms in an oligopoly market under general demand and cost functions. Each firm's optimal response to the strategies of other firms is assessed by the magnitude and sign of its conjectural variation, expressing the firm's expectation regarding the counterparty's supply quantity change in response to the firm's unit change in its supply quantity. A game of n firms with the sum of conjectural variations (SCV) regarding all counterparties as the generalized response characteristic is studied. The existence of a bifurcation of the players' response is revealed; a bifurcation is a strategy profile of the game in which both positive and negative responses are possible with an infinite-magnitude SCV value. Methods are developed for calculating the SCV value under different types of inverse demand functions (linear and power) and cost functions (linear, power, and quadratic), and the impact of these characteristics of firms on the bifurcation state is comparatively analyzed.

Keywords: oligopoly, conjectural variation, bifurcation, Stackelberg leadership

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1. INTRODUCTION

In an oligopoly game, players (firms) make assumptions about the strategies of other players (the environment) underlying their optimal response to these strategies. In the case of quantity competition, the assumptions of firms are formalized by conjectural variations [1]. This case is often considered by researchers [2] due to the preferability of quantity competition: it results in a smaller output, higher prices, and higher profits than price competition [3]. A conjectural variation (hereinafter, meaning a quantity conjectural variation) is the firm's expectation regarding the counterparty's supply quantity change in response to the firm's unit change in its supply quantity. In oligopoly theory, it is conventional to consider the optimal (consistent [4]) conjectural variation calculated from the player's necessary condition of optimality, i.e., the one corresponding to the player's best response. In other words, the player's strategy choice model (utility function) being unknown, the awareness of the conjectural variation allows predicting the player's behavior.

In addition, when assessing the conjectural variations, a player can assume that the counterparty is also assessing him, i.e., suppose the former's optimal behavior. In this case, the counterparty is called a Stackelberg leader whereas the given player a follower. However, the counterparty may argue by analogy, treating the given player as a Stackelberg leader and calculating the conjectural variation from the leader's optimal response (thereby becoming a second-level leader for the given player). This sequence of players' reasoning is called strategic reflexion. Thus, an analysis of conjectural variations inevitably leads to the problem of multilevel leadership [5]. Consequently, the vector of the conjectural variations of all players is a complex characteristic of the strategy profile of the game with these mental profiles of the firms, as conjectural variations are functions of the role of each player in the hierarchy of multilevel leadership.

In an oligopoly game of $n > 2$ players, the firm's behavior is determined by the sum of its conjectural variations (SCV) regarding all environment players. If the player's SCV is negative, then its optimal strategy is to increase the supply quantity, and vice versa. Therefore, in the n-player game, the awareness of all components of the vector of conjectural variations of all players is not necessary for predicting the game outcome: it suffices to know the components of the SCV vector of all players. No doubt, the awareness of the players' utility functions is required to determine the SCV vector; nevertheless, given available limits for typical utility functions and the nature of SCV changes, one receives an information base for predicting game outcome limits.

Typical utility functions are defined by a set of demand functions and cost functions [5–19]. In the studies of oligopoly, the most common inverse demand functions are the linear [5, 6, 9–15] and power [5, 16–19] ones. The set of functions describing the costs of oligopolists is somewhat wider: the linear function [10, 12–14, 16, 18], the power function [6, 17], and the quadratic function [5, 7–9, 11, 15, 19]. Obviously, in the vast majority of publications, researchers consider the linear models of demand and costs: in this case, it is easy to calculate conjectural variations from the best response functions (reaction functions) in explicit form. The power cost function can be either convex or concave for different degrees; a concave cost function corresponds to the positive scale effect whereas a convex cost function to the negative scale effect. The quadratic cost function is used only to describe the negative scale effect in the convex case: otherwise, the transition to a decreasing dependence of costs on output may occur, which disagrees with the economic realities.

Thus, when assessing the behavior of firms in an oligopoly game, a topical problem is to analyze the nature and limits of SCV changes due to the changes in the profile of their reflexive beliefs under different utility functions of the players.

2. FORMULATION OF THE OLIGOPOLY GAME MODEL

Consider quantity mono-product competition in an oligopoly of $n > 2$ firms. Let all firms have a common inverse demand function $P(Q)$ decreasing in the total supply quantity Q, and let the cost function $C_i(Q_i)$ of each firm i be nondecreasing in its supply quantity Q_i.

We suppose the possibility of reflexion for each player (firm), given by a reflexion rank r. The player's reflexive behavior consists in putting forward some beliefs about the strategies of its environment (the other players), which leads to the appearance of phantom players in the game [20]. In this case, reflexion rank is a numerical characteristic of such beliefs, and the sequence of reflexion ranks defines the following hierarchy of phantom players:

—At rank $r = 1$, the player is aware that the environment does not know its strategy, i.e., the other players are followers and this player becomes a first-level Stackelberg leader.

—At rank $r = 2$, according to the player's information, it is surrounded by first-level Stackelberg leaders; hence, this player becomes a second-level Stackelberg leader.

—At an arbitrary rank r, the player knows that the environment players are $(r - 1)$th-level Stackelberg leaders; therefore, this player becomes an rth-level Stackelberg leader.

Thus, the real game of firms in an oligopoly market will be treated below as an information game of phantom players, each having different leadership levels depending on the degree of its awareness. Such a situation is commonly called multilevel leadership (a multiple leader–follower game) [5], and leadership levels are given by the reflexion rank r.

A multiple leader–follower game is a tuple of the form

$$\Gamma = \langle N, \{Q_i, i \in N\}, \{\Pi_i, i \in N\}, \{r_i, i \in N\} \rangle,$$

where $N = \{1, \dots, n\}$ denotes the set of players, $\{Q_i, i \in N\}$ is their action vector (the strategy profile of the game), $\{\Pi_i, i \in N\}$ is the vector of their utility functions, and $\{r_i, i \in N\}$ is the vector of their ranks.

The utility function of player i has the form

$$\Pi_i(Q, Q_i) = P(Q)Q_i - C_i(Q_i).$$

Differentiating the utility functions of the players, we define the system of necessary conditions for Nash equilibrium:

$$P(Q) + (1 + S_i^r)Q_i P_Q^/ - C_{iQ_i}^/ = 0, \quad i \in N, \tag{1}$$

where $S_i^r = \sum_{j \in N \setminus i} Q_{j(r)Q_i}^/$ is the sum of the conjectural variations of player i at a reflexion rank r (each component $Q_{jQ_i}^{r/}$ is the conjectural variation of player i, i.e., the expected change in the quantity of player j in response to the unit quantity increase of player i); the value $Q_{jQ_i}^{r/} = \rho_{ij}^r$ is calculated by differentiating equation (1) for player j, which confirms its optimality.

An equilibrium in this game, i.e., a solution of system (1) that maximizes the utility functions $\Pi_i(Q, Q_i)$ of the players, exists under the condition established by W. Novshek [21]:

$$P_Q^/ + P_{QQ}^{//} Q < 0.$$

This condition depends on the type of demand functions: for linear and exponential demand functions, it is satisfied; for the power demand function, it fails, and the existence of an equilibrium requires the nondecreasing property of the cost functions, $C_{iQ_i}^/ \geqslant 0$.

The solution of system (1) can be found if the SCV values S_i^r are known for all players. They are calculated using the following recurrent formula [6] at an arbitrary reflexion rank:

$$S_i^r = \left(\frac{1}{\displaystyle\sum_{j \in N \setminus i} \frac{1}{u_j - S_j^{r-1} + 1}} - 1 \right)^{-1}. \tag{2}$$

Due to (2), the player's SCV depends on two characteristics of the environment players:

—the mental types of players, defined by their SCV values S_j^{r-1} at the previous reflexion rank;

—the technological types related to the type of the cost functions of the environment players, defined by the parameters u_j; for some types of demand functions (if $P_{QQQ_i}^{//} \neq 0$, see below), this parameter also describes the player's mental type.

Note that formula (2) is presented for the conjectural variations independent of the actions of players, i.e., under the condition $\rho_{ijQ_i}^/ = 0$; the more general case $\rho_{ijQ_i}^/ \neq 0$ was described in [6]. It was also demonstrated therein that conjectural variations weakly depend on the supply quantities of players, i.e., $\rho_{ijQ_i}^/ \approx 0$. Below, we will justify this premise for the demand and cost functions under considerations, showing that the SCV values and types of the demand and cost functions of the environment players have the greatest impact on the player's SCV value.

Proposition 1. *The parameter* u_i *in* (2) *is given by*

$$u_i = -1 + \frac{P_{Q_i}^/ + (1 + S_i^{r-1})Q_i P_{QQ_i}^{//} - C_{iQ_iQ_i}^{//}}{|\ P_Q^/\ |}. \tag{2a}$$

We will call u_i *the nonlinearity coefficient* since it characterizes the impact of the nonlinearity of the demand and cost functions on the type of equation (1) of player i : for $u_i = -2$, the corresponding equation of system (1) is linear.

Thus, according to (1), the computation of the game equilibrium directly depends on the SCV value. In turn, this value is predetermined by the peculiarities of the functions $P(Q)$ and $C_i(Q_i)$; see formula (2). Therefore, we will study possible SCV values under different combinations of these functions.

3. RESULTS

3.1. Methods for Calculating Conjectural Variations

Whenever the player's number i does not matter, it will be omitted below, and the player's action will be denoted by $q = Q_i \forall i \in N$. Consider the inverse demand functions

$$P_1(Q) = a - bQ, \quad a > 0, \quad b > 0, \quad a \gg b, \tag{3a}$$

$$P_2(Q) = AQ^{\alpha}, \quad A > 0, \quad \alpha < 0, \quad \mid \alpha \mid < 1, \tag{3b}$$

and the cost functions

$$C_1(q) = B_0 + B_1 q, \quad B_0 \geqslant 0, \quad B_1 > 0, \tag{4a}$$

$$C_2(q) = B_0 + B_1 q^{\beta}, \quad B_0 \geqslant 0, \quad B_1 > 0, \quad \beta \in (0, 2), \tag{4b}$$

$$C_3(q) = B_0 + B_1 q + \frac{B_2}{2} q^2, \quad B_0 \geqslant 0, \quad B_1, B_2 > 0, \tag{4c}$$

where a, b, A, and α are the constant coefficients of the demand functions and B_0, B_1, B_2, and β are the constant coefficients of the cost functions.

Here, we adopt the notations of function types: $P_k(Q)$, $k = 1, 2$, is the demand function of type k ($k = 1$ corresponds to the linear function and $k = 2$ to the power function); $C_m(q)$, $m = 1, 2, 3$, is the cost function of type m ($m = 1$ corresponds to the linear function, $m = 2$ to the power function, and $m = 3$ to the quadratic function).

Using formula (2), we derive expressions for $P^{/}_{Q}$, $P^{/}_{q}$, $P^{//}_{Qq}$, and $C^{//}_{qq}$ in the case of the functions (3) and (4):

$$P^{/}_{1_Q} = P^{/}_{1_q} = -b, \quad P^{//}_{1_{Qq}} = 0, \tag{5a}$$

$$P^{/}_{2_Q} = P^{/}_{2_q} = A\alpha Q^{\alpha-1}, \quad P^{//}_{2_{Qq}} = A\alpha(\alpha - 1)Q^{\alpha-2}, \tag{5b}$$

$$C^{//}_{1_{qq}} = 0, \quad C^{//}_{2_{qq}} = B_1\beta(\beta - 1)q^{\beta-2}, \quad C^{//}_{3_{qq}} = B_2. \tag{5c}$$

As a result, the parameters $u_{km},\ k = 1, 2,\ m = 1, 2, 3$, of the functions (3) and (4) are given by

$$u_{11} = -2, \quad u_{21} = -2 + (1 + S^{r-1})(1 - \alpha)\frac{q}{Q}, \tag{6a}$$

$$\begin{gathered} u_{12} = -2 - \frac{B_1}{b}\beta(\beta - 1)q^{\beta-2}, \\ u_{22} = -2 + (1 + S^{r-1})(1 - \alpha)\frac{q}{Q} - \frac{B_1}{A \mid \alpha \mid Q^{\alpha-1}}\beta(\beta - 1)q^{\beta-2}, \end{gathered} \tag{6b}$$

$$u_{13} = -2 - \frac{B_2}{b}, \quad u_{23} = -2 + (1 + S^{r-1})(1 - \alpha)\frac{q}{Q} - \frac{B_2}{A \mid \alpha \mid Q^{\alpha-1}}. \tag{6c}$$

Note that the parameter B_1 in (6) corresponds only to the case of the power function.

3.2. Comparative Analysis of Conjectural Variations

With the notation $s_i^r = \sum_{j\in N\backslash i} \frac{1}{u_j - S_j^{r-1}+1}$, formula (2) is simplified to

$$S_i^r = \left(\frac{1}{s_i^r} - 1\right)^{-1}, \tag{7}$$

where s_i^r expresses the *aggregate* of the cost functions and SCV values of the environment of player i, i.e., a generalized characteristic of the technological and mental types of the other players.

Due to formula (7), the function $S_i^r(u_j, S_j^{r-1})$ suffers from a discontinuity of the second kind (Fig. 1) under the condition

$$s_i^r = \sum_{j\in N\backslash i} \frac{1}{u_j - S_j^{r-1} + 1} = 1; \tag{7a}$$

moreover, it takes infinitely large positive and negative values as $s_i^r \to 1 - 0$ and $s_i^r \to 1 + 0$, respectively. The function $s_i^r(u_j, S_j^{r-1})$ has a discontinuity of the second kind for $S_j^{r-1} = u_j + 1 \forall j \in N\backslash i$, which does not cause a discontinuity of the function $S_i^r(u_j, S_j^{r-1})$.

The discontinuity of the second kind of the function $S_i^r(u_j, S_j^{r-1})$ means that at the point of discontinuity $(u_j^0, S_j^{0,r-1})$, $j \in N\backslash i$, player i at a reflexion rank r can simultaneously have two SCV values ($+\infty$ and $-\infty$). Let us consider the sequential reflexion of the players regarding each other's behavior as a dynamic process on the numerical sequence of ranks $r = 1, 2, \ldots$. Then, by analogy with the solutions of some differential equations, we can say that there is a bifurcation of the player's beliefs. In this case, the *bifurcation state* of the beliefs of player i is a combination of the

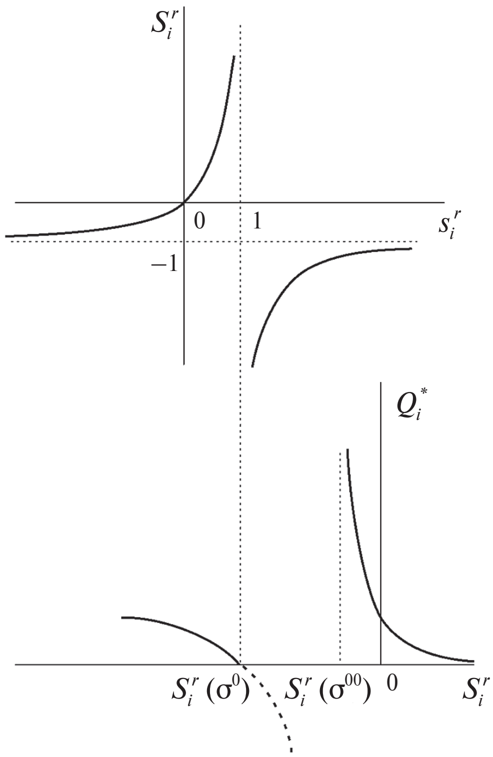

Fig. 1. The SCV value of player i depending on the aggregate of the cost functions and SCV values of the environment (top) and the equilibrium action of player i depending on SCV value (bottom).

technological types of the environment players, defined by their cost functions, and the mental types of the environment players, expressed by their leadership levels (numerically defined in the SCV form), under which player i can simultaneously expect infinitely large, both positive and negative, reactions (SCV values) of the environment.

Function (7) (see the upper part of Fig. 1) allows estimating the following intervals of SCV changes:

$$S_i^r = \begin{cases} \in [-1, 0) & \text{if } s_i^r \leqslant 0 \\ \in (0, \infty) & \text{if } 0 < s_i^r < 1 \\ \in (-\infty, -1) & \text{if } s_i^r < -1. \end{cases} \tag{7b}$$

Hence, by characterizing the dependence of the aggregate s_i^r on the environment's nonlinearity coefficients u_j, which depend on the types of the demand and cost functions and are low-sensitive to the supply quantities, and on the environment's SCV value S_j^{r-1}, which depends on the latter's mental type, we can estimate the impact of these parameters on the player's SCV value.

To qualitatively analyze in comparative terms the impact of the types of the functions $P_k(Q)$ and $C_m(q)$ and study the bifurcation phenomenon, we consider the case of identical players. Assume that for all environment players, the nonlinearity coefficients and SCV values are the same: $u_j = u$ $\forall j \in N$, $S_j^{r-1} = \sigma$ $\forall j \in N$. In this case,

$$S_i^r = \frac{n-1}{u+2-\sigma-n}, \quad s_i^r = \frac{n-1}{u-\sigma+1}. \tag{8}$$

Formulas (7a) and (8) lead to the following result.

Proposition 2. *If $u_j = u\,\forall j \in N$ and $S_j^{r-1} = \sigma\,\forall j \in N$, then the SCV function $S_i^r(u, \sigma)$ has the following properties:*

i) *a discontinuity of the second kind at $\sigma = \sigma^0 = u + 2 - n$ (except for the case of the linear demand and cost functions) and*

$$\lim_{\sigma \to (u+2-n)-0} S_i^r = \infty, \quad \lim_{\sigma \to (u+2-n)+0} S_i^r = -\infty; \tag{8a}$$

ii) *values belonging to the intervals*

$$S_i^r \begin{cases} \in (0, \infty) & \text{if } \sigma \in (-\infty, \sigma^0) \\ \in (-\infty, 0) & \text{if } \sigma \in (\sigma^0, \infty) \end{cases} \tag{8b}$$

(*except for the case of the linear demand and cost functions, in which $S_i^r \in [-1, 0)$*).

A clear illustration of a bifurcation follows from the explicit-form solution of the system of equilibrium equations (1), known for the case of the linear demand and cost functions. However, in this case, infinite values of conjectural variations do not arise (they are bounded by the range $(-1, 0]$). For power cost functions, an explicit-form solution does not exist [6], so we consider the case of the linear demand function and the quadratic cost functions.

Proposition 3. *In the case of the linear demand function and the quadratic cost functions, the general solution of game* (1) *has the form*

$$Q_i^* = \frac{D_i \left[\prod_{j=1 \backslash i}^{n} \left(\gamma_j^r - 1 \right) + \sum_{j=1 \backslash i}^{n} \prod_{\mu=1 \backslash j,i}^{n} \left(\gamma_\mu^r - 1 \right) \right] - \sum_{j=1 \backslash i}^{n} \left[D_j \prod_{\mu=1 \backslash i,j}^{n} \left(\gamma_\mu^r - 1 \right) \right]}{\prod_{j=1}^{n} \left(\gamma_j^r - 1 \right) + \sum_{j=1}^{n} \prod_{\mu=1 \backslash j}^{n} \left(\gamma_\mu^r - 1 \right)}; \tag{9}$$

in the particular case $n = 3$, the formula reduces to

$$Q_i^* = \frac{D_i\left(\prod\limits_{j=1\backslash i}^{3}\gamma_j^r - 1\right) - \sum\limits_{j=1\backslash i}^{3}\prod\limits_{\mu=1\backslash i}^{3} D_j\gamma_\mu^r + \sum\limits_{j=1\backslash i}^{3} D_j}{\gamma_1^r\gamma_2^r\gamma_3^r - \gamma_1^r - \gamma_2^r - \gamma_3^r + 2}; \tag{9a}$$

in the particular case $n = 3$ with the identical types of the players, $D = D_i\ \forall i \in N$, the formula becomes

$$Q_i^* = D\frac{\prod\limits_{j=1\backslash i}^{3}\gamma_j^r - \sum\limits_{j=1\backslash i}^{3}\gamma_j^r + 1}{\gamma_1^r\gamma_2^r\gamma_3^r - \gamma_1^r - \gamma_2^r - \gamma_3^r + 2}, \tag{9b}$$

where

$$D_i = \frac{a - B_{1i}}{b}, \quad \gamma_i = 2 + S_i^r + \frac{B_{2i}}{b},$$

and the symbol "" indicates the game equilibrium.*

Under a *belief bifurcation*, two cases are simultaneously possible. They will be described based on (9b) for the identical (same-type) environment players, see the corresponding condition in Proposition 2. Then, considering (6c), we have $\gamma_j = \gamma = -u_j + S_j^{r-1} = -u + \sigma$, $j = 2, 3$, and $\gamma < 0$ for $\sigma = \sigma^0$ since $B_2 > 0$.

The first case $S_i^r \to \infty$ means that for the environment players, the optimal strategy is infinite growth of the supply quantity, limited by the parameters of the demand function $P(Q)$ and the technological capabilities of the firms. (Recall that conjectural variations are considered as strategies.) In this case, if the environment's SCV values S_j^r, $j = N\backslash i$, are finite numbers, then by (9b) the player's optimal response vanishes on the right, i.e., $Q_i^{r^*} \to 0 + 0$. In other words, the player seeks to reduce the supply quantity to zero.

The second case $S_i^r \to -\infty$ implies an infinite reduction in the supply quantity by the environment players, although they can actually reduce the supply only to zero. Due to (9b), the player's optimal response vanishes on the left, i.e., $Q_i^{r^*} \to 0 - 0$. This can be interpreted as the player's largest acceptable response to the negative value of the total supply quantity predicted by this player based on $S_i^r \to -\infty$.

Interestingly, a belief bifurcation should lead to an *action bifurcation* at the subsequent reflexion ranks. This fact can also be demonstrated from the optimal SCV formula (8) and the equilibrium action (9b).

If $S_i^r \to \infty$, at the next reflexion rank $(r + 1)$, we consider the situation from the environment's viewpoint (i.e., as $\sigma \to \infty$); from (8) it follows that $S_j^{r+1} \to 0$. Returning to player i at rank $(r + 2)$, for which $\sigma \to 0$, from (8) we also obtain $\lim_{\sigma\to 0} s_i^{r+2} = \frac{n-1}{u-\sigma+1} < 0$ since $u < -2$ by (6c). Consequently, $S_i^{r+2} \to -1$ as $\sigma \to \sigma^{00} = u + 1$, and when preserving the environment's responses by the type $S_j^{r+1} \to 0$, formula (9b) implies $Q_1^* \to \infty$ ($\gamma = 1$ and $Q_1^* = D\frac{\gamma^2-2\gamma+1}{\gamma_1(\gamma^2-1)-2(\gamma-1)}$). Thus, the SCV-defined mental response bifurcation leads to an equilibrium bifurcation in the game. These considerations are illustrated in Fig. 1 (the lower part).

The case of identical players is the basis for comparatively analyzing the impact of the types of demand and cost functions on the SCV value. According to (8a), the bifurcation point σ^0 shifts upwards when increasing the nonlinearity coefficients u of the environment players and behaves oppositely when decreasing u decreases. In other words, a bifurcation state occurs under higher values of the environment's SCV value. Due to conditions (8b), if the nonlinearity coefficients are larger, the environment's SCV value should be larger so that S_i^r belongs to the corresponding ranges.

Let us characterize the dependence of the SCV value of player i on the nonlinearity coefficient u_l of some environment player l as well as on the environment's SCV value and q, Q.

Proposition 4. *The SCV value S_i^r of player i at a reflexion rank r has the following properties:*

i) *goes down when increasing the nonlinearity coefficient u_l of environment player l and up when increasing S^{r-1} :*

$$S_{iul}^{r/} < 0,\ S_{iS^{r-1}}^{r/} > 0; \tag{10a}$$

ii) *in the case of the linear demand function, is independent of q under the linear and quadratic cost functions of the environment, goes down (up) when increasing q for $\beta > 1$ (for $\beta < 1$, respectively) under the power cost functions of the environment, and is independent of Q under any cost functions of the environment:*

$$S_{i\ q}^{r/}\Big|_{\substack{k=1\\m=1,3}} = S_{i\ Q}^{r/}\Big|_{\substack{k=1\\m=1,2,3}} = 0,\quad S_{i\ q}^{r/}\Big|_{\substack{k=1\\m=2}} \begin{cases} < 0 & \text{for } \beta > 1,\\ > 0 & \text{for } \beta < 1; \end{cases} \tag{10b}$$

iii) *in the case of the power demand function, goes down (up) when increasing q if $S^{r-1} > -1$ (if $S^{r-1} < -1$, respectively) under the linear and quadratic cost functions of the environment, down if $S^{r-1} > -1$ under the convex power cost functions ($\beta > 1$) and under the concave power cost functions ($\beta < 1$) provided that $\varphi < 1$, and up (down) if $S^{r-1} < -1$ under the concave power cost functions (under the convex power cost functions provided that $\varphi < -1$, respectively);*

goes up (down) when increasing Q if $S^{r-1} > -1$ (if $S^{r-1} < -1$, respectively) under the linear cost functions and the quadratic cost functions (in the latter case, goes down provided that $\zeta < 1$), up (down) if $S^{r-1} > -1$ under the convex power cost functions ($\beta > 1$) (under the concave power cost functions ($\beta < 1$) provided that $\varphi < 1$, respectively), and down (up) if $S^{r-1} < -1$ under the concave power cost functions (under the convex power cost functions provided that $\psi > -1$, respectively):

$$\begin{gathered}
S_{i\ q}^{r/}\Big|_{\substack{k=2\\m=1,3}} \begin{cases} < 0 & \text{for } S^{r-1} > -1,\\ > 0 & \text{for } S^{r-1} < -1, \end{cases}\\
S_{i\ Q}^{r/}\Big|_{\substack{k=2\\m=1}} \begin{cases} > 0 & \text{for } S^{r-1} > -1,\\ < 0 & \text{for } S^{r-1} < -1, \end{cases}\\
S_{i\ q}^{r/}\Big|_{\substack{k=2\\m=2}} \begin{cases} < 0 & \text{if } \varphi < 1 \text{ for } t = 1,\\ < 0 & \text{for } t = 2,\\ > 0 & \text{for } t = 3,\\ < 0 & \text{if } \varphi < -1 \text{ for } t = 4, \end{cases}\\
S_{i\ Q}^{r/}\Big|_{\substack{k=2\\m=2}} \begin{cases} < 0 & \text{if } \psi > 1 \text{ for } t = 1,\\ > 0 & \text{for } t = 2,\\ < 0 & \text{for } t = 3,\\ < 0 & \text{if } \psi > -1 \text{ for } t = 4, \end{cases}\\
S_{i\ Q}^{r/}\Big|_{\substack{k=2\\m=3}} \begin{cases} > 0 & \text{for } S^{r-1} > -1,\\ < 0 & \text{if } \zeta < 1 \text{ for } S^{r-1} < -1; \end{cases}
\end{gathered} \tag{10c}$$

iv) *weakly depends on the supply quantities of the players compared to the impact of the environment's SCV value:*

$$S_{i\ q}^{r/} \ll S_{i\ S^{r-1}}^{r/}, \tag{10d}$$

where

$$\varphi = \frac{B_1\beta(1-\beta)(2-\beta)q^{\beta-3}}{A|\alpha||1+S^{r-1}|(1-\alpha)Q^{\alpha-2}}, \quad \psi = \varphi\frac{1-\alpha}{2-\beta}, \quad \zeta = \frac{B_2 Q^{2-\alpha}}{A|\alpha||1+S^{r-1}|q},$$

and the additional notations are

$$t = 1:\ S^{r-1} > -1 \wedge \beta < 1, \qquad t = 2:\ S^{r-1} > -1 \wedge \beta > 1,$$
$$t = 3:\ S^{r-1} < -1 \wedge \beta < 1, \qquad t = 4:\ S^{r-1} < -1 \wedge \beta > 1.$$

Now we compare the bifurcation points under different demand and cost functions.

Proposition 5. *Under different types of the demand and cost functions of the environment, the bifurcation point σ^0 satisfies the following relations:*

$$\sigma^0_{23} > \sigma^0_{21}, \tag{11a}$$
$$\sigma^0_{22} > \sigma^0_{21} \text{ for } \beta > 1, \tag{11b}$$
$$\sigma^0_{21} > \sigma^0_{12} \text{ for } B_1 > \bar{B}_1, \tag{11c}$$
$$\sigma^0_{21} > \sigma^0_{13} \text{ for } B_2 < \bar{B}_2, \tag{11d}$$
$$\sigma^0_{12} > \sigma^0_{22} \text{ for } B_1 < \bar{\bar{B}}_1, \text{ if } \beta > 1 \text{ or for } B_1 > \bar{\bar{B}}_1 \text{ if } \beta < 1, \tag{11e}$$
$$\sigma^0_{13} < \sigma^0_{23} \text{ for } B_2 < \bar{\bar{B}}_2, \tag{11f}$$
$$\sigma^0_{12} > \sigma^0_{13} \wedge \sigma^0_{22} > \sigma^0_{23} \text{ for } \beta > 1 \text{ and } \frac{B_1}{B_2} > \frac{1}{\lambda}, \tag{11g}$$

where

$$\bar{B}_1 = b\frac{\delta}{\lambda}, \quad \bar{B}_2 = b\delta, \quad \bar{\bar{B}}_1 = \frac{\delta}{\lambda}(\chi - b), \quad \bar{\bar{B}}_2 = \delta\frac{\chi b}{\chi - b},$$
$$\delta = (1+S^{r-1})(1-\alpha)\frac{q}{Q}, \quad \chi = A|\alpha|Q^{\alpha-1} > 0, \quad \lambda = \beta(\beta-1)q^{\beta-2}.$$

The corresponding relations for the bifurcation point under the other types of the demand and cost functions of the environment are presented in the Appendix.

4. FINDINGS

According to Proposition 2, a belief bifurcation occurs for a player when the environment's SCV value increases from $\sigma = u + 2 - n - 0$ to $\sigma = u + 2 - n + 0$; furthermore, *a greater magnitude of the SCV value is required to destabilize the equilibrium in the case of more players* since the bifurcation point decreases with increasing n.

Proposition 4 reveals the following major factors affecting the player's SCV value. First, there is the factor of the market-technological conditions of the game, determined by the nonlinearity coefficient: the greater the nonlinearity coefficient of the environment players is, the smaller the SCV value will be (i.e., the greater its magnitude will be). As a rule, the SCV value is negative, and *the growing magnitude of the SCV value indicates enhancing the player's response*. Therefore, the combinations of the demand and cost functions resulting in higher values of the nonlinearity coefficient contribute to enhancing the player's response. In particular, these include games with quadratic cost functions or power cost functions with the positive scale effect ($\beta < 1$), which lead to greater values of the nonlinearity coefficient compared to the linear cost model regardless of the demand model; for details, see the Appendix.

Second, *symmetric response* is observed for the players, i.e., the greater the environment's SCV value is, the greater the player's SCV value will be. This player response consonance, qualitatively

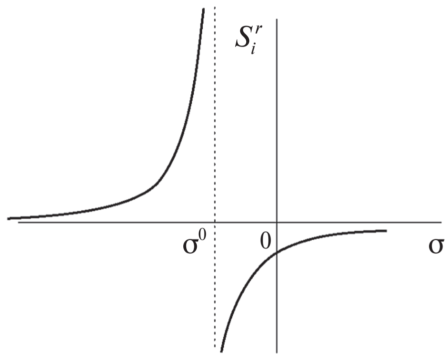

Fig. 2. The player's SCV value depending on the environment's SCV value.

illustrated in Fig. 2, is expressed as a two-step process. If the environment has negative SCV values, then increasing them (decreasing their magnitudes) is accompanied by a growth in the positive SCV value of the player ($S_i^r \to \infty$). In economic terms, the player expects expansion of the environment and, as a result, will reduce the supply quantity to zero according to formula (9b). Then a bifurcation point occurs, and the process changes in the opposite direction: applying formula (8) to the environment yields $S_j^{r+1} = \frac{n-1}{u+2-S_i^r-n}$ and, consequently, $S_j^{r+1} \to 0$. In other words, the environment expects the player's zero reaction. But in response to this situation (Fig. 2), the player's SCV value again increases, i.e., $S_j^{r+2} \to -1$, motivating the player to increase the supply quantity.

Third, the greatest impact on the player's SCV value is exerted by the environment's SPV values and types of the demand and cost functions. Despite that the player's SCV value depends on its supply quantity and the total supply quantity of all players through the nonlinearity coefficients of the environment (for the nonlinear functions), this impact is negligibly small compared to the impact of the mental types of the environment conditioned by its leadership levels.

Proposition 5 identifies the following properties of a bifurcation.

The case of *the power demand function* leads to a greater value of the bifurcation point under the quadratic cost function ($k = 2, m = 3$) compared to the linear cost function ($k = 2, m = 1$) since the quadratic function models the negative scale effect. Similarly, in the case of the power demand function, the bifurcation point under the power cost function ($k = 2, m = 2$) with the negative scale effect ($\beta > 1$) exceeds the corresponding point under the linear cost function ($k = 2, m = 1$); the converse situation occurs given the positive scale effect.

Compared to the linear demand function and different cost functions ($m = 2, 3$), the case of *the power demand function and the linear cost function* ($k = 2, m = 1$) leads to an increase in the bifurcation point under certain values of the coefficients B_1 and B_2 :

i) if $\bar{B}_1 < 0$ (i.e., given the positive scale effect for $S^{r-1} < -1$ and the negative scale effect for $S^{r-1} > -1$) since

$$\bar{B}_1 = \frac{b(1+S^{r-1})(1-\alpha)}{\beta(\beta-1)Qq^{\beta-3}} \begin{cases} > 0 \text{ if } (\beta > 1 \wedge S^{r-1} > -1) \vee (\beta < 1 \wedge S^{r-1} < -1) \\ < 0 \text{ if } (\beta < 1 \wedge S^{r-1} < -1) \vee (\beta > 1 \wedge S^{r-1} > -1); \end{cases}$$

and the scale effect as the technological type of players is opposite to the impact of the environment's SCV value as the mental type of players;

ii) if $\bar{B}_1 > 0$ and $B_1 > \bar{B}_1$, i.e., for a high growth rate of the power function;

iii) if $\bar{B}_2 > 0$, i.e., for $S^{r-1} > -1$ since $\bar{B}_2 > 0 = b(1+S^{r-1})(1-\alpha)\frac{q}{Q}$.

Compared to the *quadratic cost functions* ($m = 3$) with any demand functions ($k = 1, 2$), the case of the *power cost functions* ($m = 2$) gives the following relations for the bifurcation point:

i) The bifurcation point under the power cost functions is greater if the scale effect is negative ($\beta > 1$), for $B_2 \ll B_1$, since $\frac{B_1}{B_2} > \frac{1}{\lambda}$ implies $B_1\beta(\beta-1)q^{\beta-2} > B_2$, and $\beta(\beta-1)q^{\beta-2} \ll 1$.

ii) The bifurcation point under the power cost functions is smaller if the scale effect is positive ($\beta < 1$) because, in this case, $B_1\beta(\beta-1)q^{\beta-2} < 0$.

The case of the *quadratic cost functions* ($m = 3$) with the linear ($k = 1$) and power ($k = 2$) demand functions demonstrates two possibilities in which enhancing the environment's response compensates for the nonlinearity impact of the demand function:

i) The bifurcation point under the power cost functions is greater if $S^{r-1} > -1$ and B_2 is sufficiently small ($B_2 < \bar{\bar{B}}_2$) since $\bar{\bar{B}}_2 = \frac{b(1+S^{r-1})(1-\alpha)A|\alpha|qQ^{\alpha-2}}{A|\alpha|Q^{\alpha-1}-b} > 0$ if $A \gg b$.

ii) The bifurcation point under the power cost functions is smaller if $S^{r-1} > -1(\bar{\bar{B}}_2 < 0)$.

5. CONCLUSIONS

The market-technological conditions of an oligopoly game are described by a combination of the market demand function and the players' cost functions, which together determine their utility functions. Based on the analysis of the variety of such combinations arising in different applied problems of oligopoly modeling, this study has demonstrated the importance of the market-technological conditions of an oligopoly game for the stability of the game equilibrium. As has been established, the reason of destabilizing equilibrium, or a bifurcation of the players' actions, is a bifurcation of their beliefs: under a definite constellation of the beliefs of environment players, the player can evaluate their optimal reaction as positive and negative simultaneously. In turn, the specified constellation of players' beliefs is predetermined by their Stackelberg leadership levels and expressed by some SCV value of the environment, which can be called a bifurcation point.

The bifurcation point depends on the number of players and the nonlinearity coefficient of their utility functions, and the nonlinearity coefficient is determined by the types of the demand and cost functions. If the bifurcation point is greater for a particular combination of the demand and cost functions of the players (i.e., the SCV value of the environment has a smaller magnitude), the game situation will be more sensitive to changes in the mental types of the players. In other words, the game equilibrium can be easier destabilized in dynamics.

It is characteristic that the equilibrium cannot be destabilized under the linear demand and cost functions. Therefore, gradual changes of the equilibrium actions will be observed in real oligopoly games with the linear dependencies of the market-technological parameters, a phenomenon often encountered in practice.

APPENDIX

Proof of Proposition 1. The parameter u_i in [6] is the component of the second-order condition for the optimum of the player's utility function[2], i.e., $\Pi^{//}_{iQ_iQ_i} = u_i - S_i^{r-1} < 0$. Based on (1), we write this condition as $P^{/}_{Q_i} + (1 + S_i^{r-1})P^{/}_Q + (1 + S_i^{r-1})Q_iP^{//}_{QQ_i} - C^{//}_{iQ_iQ_i} < 0$; in view of $P^{/}_Q < 0$, this inequality can be divided by $|P^{/}_Q|$: $\frac{P^{/}_{Q_i}}{|P^{/}_Q|} - 1 - S_i^{r-1} + \frac{(1+S_i^{r-1})Q_iP^{//}_{QQ_i}}{|P^{/}_Q|} - \frac{C^{//}_{iQ_iQ_i}}{|P^{/}_Q|} < 0$, which finally yields (2a).

---

[2] In [6], this parameter has the form $u_i = -2 - \frac{C^{//}_{iQ_iQ_i}}{b}$ since it was derived under the linear demand function, for which $P^{/}_Q = P^{/}_{Q_i} = -b$.

Proof of Proposition 3. Given (3a) and (4c), equations (1) take the form

$$a - bQ - b(1 + S_i^r)Q_i - B_{2i}Q_i - B_{1i} = 0,$$

or

$$\gamma_i Q_i + \sum_{j=N\backslash i} Q_j q_{-i} = D_i, \quad i \in N;$$

solving this system by Cramer's rule gives (9).

Proof of Proposition 4. By denoting $z_j^{r-1} = \frac{1}{u_j - S_j^{r-1} + 1}$ and differentiating the expression (7), we obtain

$$S_{i\ u_l}^{r/} = \left(\frac{1}{S_i^r} - 1\right)(s_i^r)^{-2}\left(z_j^{r-1}\right)_{u_l}^{/} = -\left(1 - s_i^r\right)^{-2}\left(u_l - S_l^{r-1} + 1\right)^{-2} < 0, \tag{A.1}$$

$$S_{i\ q}^{r/} = S_{i\ u_l}^{r/} u_{l\ q}^{/}, \quad S_{i\ Q}^{r/} = S_{i\ u_l}^{r/} u_{l\ Q}^{/}, \quad S_{i\ S^{r-1}}^{r/} > 0, \quad S_{i\ u_l}^{r/} < 0. \tag{A.2}$$

To simplify the analysis of formulas (6), let us introduce the notations

$$\delta = (1 + S^{r-1})(1 - \alpha)\frac{q}{Q}\begin{cases} > 0 & \text{for } S^{r-1} > -1 \\ < 0 & \text{for } S^{r-1} < -1, \end{cases} \quad \chi = A|\alpha|Q^{\alpha-1} > 0, \tag{A.3}$$

$$\lambda = \beta(\beta - 1)q^{\beta-2}\begin{cases} < 0 & \text{for } \beta < 1 \\ > 0 & \text{for } \beta > 1. \end{cases}$$

In addition, with the compact notations x, y, z, X, Y, and Z for $u_{km}, k = 1, 2, m = 1, 2, 3$, formulas (6) reduce to

$$x = u_{11} = -2, \quad X = u_{21} = -2 + \delta, \tag{A.4a}$$

$$y = u_{12} = -2 - \frac{B_1}{b}\lambda, \quad Y = u_{22} = -2 + \delta - \frac{B_1}{\chi}\lambda, \tag{A.4b}$$

$$z = u_{13} = -2 - \frac{B_2}{b}, \quad Z = u_{23} = -2 + \delta - \frac{B_2}{\chi}. \tag{A.4c}$$

Analysis of (A.3) and (A.4) shows the existence of four possible cases depending on the values of the parameters β and S^{r-1}, further indicated by the symbol t: 1) $t = 1 : S^{r-1} > -1 \wedge \beta < 1$; in this case, $\delta > 0 \wedge \lambda < 0$; 2) $t = 2 : S^{r-1} > -1 \wedge \beta > 1$; in this case, $\delta > 0 \wedge \lambda > 0$; 3) $t = 3 : S^{r-1} < -1 \wedge \beta < 1$; in this case, $\delta < 0 \wedge \lambda < 0$; 4) $t = 4 : S^{r-1} < -1 \wedge \beta > 1$; in this case, $\delta < 0 \wedge \lambda > 0$.

Differentiating (A.4) yields

$$\begin{gathered} x_q^{/} = z_q^{/} = 0, \quad X_q^{/} = Z_q^{/} = \frac{\delta}{q}, \\ y_q^{/} = -\frac{B_1\lambda}{bq}(\beta - 2), \quad Y_q^{/} = \frac{\delta}{q} - \frac{B_1\lambda}{\chi q}(\beta - 2), \end{gathered} \tag{A.5}$$

$$\begin{gathered} x_Q^{/} = y_Q^{/} = z_Q^{/} = 0, \quad X_Q^{/} = -\frac{\delta}{Q}, \\ Y_q^{/} = -\frac{\delta}{Q} - \frac{B_1\lambda}{\chi q}(\beta - 2), \quad Z_Q^{/} = -\frac{\delta}{Q} - \frac{B_2}{\chi Q}(1 - \alpha). \end{gathered} \tag{A.6}$$

Due to (A.2) and (A.3), from these formulas we obtain the following results:

1) under the linear demand function ($k = 1$),

$$S_{i\ q}^{r/}\Big|_{\substack{k=1\\ m=1,3}} = S_{i\ Q}^{r/}\Big|_{\substack{k=1\\ m=1,3}} = 0, \quad S_{i\ q}^{r/}\Big|_{\substack{k=1\\ m=2}} \begin{cases} < 0 & \text{for } \beta > 1 \\ > 0 & \text{for } \beta < 1, \end{cases}$$

2) under the power demand function ($k = 2$), $Y_q^/ > 0$, i.e., due to (A.1), $S_{i\ q}^{r/}\Big|_{\substack{k=2\\ m=2}} < 0$ if $\delta + \frac{B_1\lambda}{\chi}(2-\beta) > 0$; this inequality leads to the four possible cases: for $t = 1$, the inequality $1 > -\frac{B_1\lambda}{\chi\delta}(2-\beta)$ is valid, and the substitution of (A.3) gives $1 > \varphi = \frac{B_1\beta(1-\beta)(2-\beta)q^{\beta-3}}{A|\alpha||1+S^{r-1}|(1-\alpha)Q^{\alpha-2}}$; for $t = 2$, the inequality is the same and $\varphi < 0$, i.e., $Y_q^/ > 0$ without additional conditions; for $t = 3$, we have $\delta + \frac{B_1\lambda}{\chi}(2-\beta) < 0$, and consequently, $Y_q^/ < 0$; for $t = 4$, $Y_q^/ > 0$ if $\varphi < -1$; the derivatives $Y_Q^/ > 0$ and $Z_Q^/ > 0$ are considered by analogy.

Let us compare $S_{i\ S^{r-1}}^{r/}$ and $S_{i\ q}^{r/}$ by magnitude, observing that

$$S_{i\ S^{r-1}}^{r/} = (1-s_i^r)^{-2}(u_l - S_l^{r-1} + 1)^{-2}, \quad S_{i\ q}^{r/} = (1-s_i^r)^{-2}(u_l - S_l^{r-1} + 1)^{-2} u_{l\ q}^/.$$

From (A.5) it follows that

$$u_{11q}^/ = u_{13q}^/ = 0, \quad u_{21q}^/ = u_{23q}^/ = (1+S^{r-1})(1-\alpha)\frac{1}{Q},$$
$$u_{12q}^/ = \frac{B_1}{b}(\beta-2)\beta(\beta-1)q^{\beta-3},$$
$$u_{22q}^/ = (1+S^{r-1})(1-\alpha)\frac{1}{Q} - \frac{B_1}{A|\alpha|Q^{\alpha-1}}(\beta-2)\beta(\beta-1)q^{\beta-3}.$$

Obviously, $\lim_{q\to\infty} u_{l\ q}^/ \to 0$, and therefore, $S_{i\ q}^{r/} \ll S_{i\ S^{r-1}}^{r/}$.

Proof of Proposition 5. For the linear and quadratic cost functions with any parameter values, we have the relations

$$x > z, \quad X > Z.$$

In the other cases, the nonlinearity coefficients satisfy the following relations: $x < X$ for $S^{r-1} > -1$; $x < y$ for $\beta < 1$; $x < Y$ for $B_1 < \bar{\bar{\bar{B}}}_1$; $x < Z$ for $B_2 < \bar{\bar{\bar{B}}}_2$; $X < Y$ for $\beta > 1$; $X < y$ for $B_1 > \bar{B}_1$; $X < z$ for $B_2 < \bar{B}_2$; $y < Y$ for $B_1 < \bar{\bar{B}}_1$ if $\beta > 1$, or for $B_1 > \bar{\bar{B}}_1$ if $\beta < 1$; $z < Z$ for $B_2 < \bar{\bar{B}}_2$; $y < z$ for $\beta > 1$ and $\frac{B_1}{B_2} > \frac{1}{\lambda}$; $Y < Z$ for $\beta > 1$ and $\frac{B_1}{B_2} > \frac{1}{\lambda}$; $Y < Z$ for $\beta > 1$ and $\frac{B_1}{B_2} > \frac{1}{\lambda}$, where $\bar{\bar{\bar{B}}}_1 = \delta\frac{\chi}{\lambda}$, $\bar{B}_1 = b\frac{\delta}{\lambda}$, $\bar{\bar{\bar{B}}}_2 = \delta\chi$, $\bar{B}_2 = b\delta$, $\bar{\bar{B}}_1 = \frac{\delta}{\lambda}(\chi - b)$, and $\bar{\bar{B}}_2 = \delta\frac{\chi b}{\chi - b}$. Due to (10a), greater values of u_{km} lead to smaller values of S_{ikm}^r. Therefore, these relations yield the desired inequalities for σ_{km}^0.

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